Hawking Radiation – Revisited

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Contents

- Recap along the line of Hawking-Wald (Geroch) ; Update

- The Hilbert space of a free quantum field

$$\mathbf{F} = \mathbf{H}_0 \oplus \mathbf{H}_1 \oplus \mathbf{H}_2 \oplus \ldots \tag{1}$$

where $\mathbf{H}_1 = \mathbf{H}$ and $\mathbf{H}_n = \otimes_n \mathbf{H}$.

- The Hilbert space of a free quantum field

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where $\mathbf{H}_1 = \mathbf{H}$ and $\mathbf{H}_n = \otimes_n \mathbf{H}$.

- Suppose the 1-particle Hilbert space $\mathbf{H}$ is separable, having an orthonormal basis e_i. Then a typical Fock space element is

$$\begin{aligned}\Psi &= (\Psi_0, \Psi_1, \Psi_2, \Psi_3, \ldots) \\ &= (\psi_0,\ \psi_i e_i,\ \frac{1}{2!}\psi_{ij} e_i \otimes e_j,\ \frac{1}{3!}\psi_{ijk} e_i \otimes e_j \otimes e_k, \ldots)\end{aligned} \tag{2}$$

where $\psi_0, \psi_i, \psi_{ij}, \ldots$ are arbitrary complex numbers that are totally symmetric for the bosonic states.

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- (4) Bosonic case: Both $a_i, a_i^\dagger$ are derivations on tensors. For a Fock space element Ψ,

$$a_i \Psi = (\psi_i,\ \psi_{ik} e_k,\ \frac{1}{2!}\psi_{ikl} e_k \otimes e_l,\ \ldots) \tag{3}$$

$$a_i^\dagger \Psi = (0, \psi_0 e_i,\ \psi_k e_{(i} \otimes e_{k)},\ \frac{1}{2!}\psi_{kl} e_{(i} \otimes e_k \otimes e_{l)},\ \ldots) \tag{4}$$

where $e_{(i} \otimes e_{j)} = \frac{1}{2}(e_i \otimes e_j + e_j \otimes e_i)$.

- It is straightforward to verify that

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a_i a_j &= a_j a_i, \\
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N\Psi = N(\Psi_0, \Psi_1, \Psi_2, \Psi_3, ...) = (0\Psi_0, 1\Psi_1, 2\Psi_2, 3\Psi_3, ...) \tag{6}
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- Also, $[N, a_i] = -a_i$ and $[N, a_i^\dagger] = a_i^\dagger$.

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- Lesson: Creation and annihilation operators exist on an arbitrary Fock space irrespective of the details of how the 1-particle Hilbert space is constructed.
- I have demonstrated it for the bosonic case. A similar construction exists for the fermionic case also.

- A massless free scalar field (in an arbitrary spacetime) is a hermitian operator ϕ on the Fock space that is distribution valued.

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- All solutions of the field equation form a complex vector space **S**. A scalar product in **S** is defined by

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- (3) It is hermitian, $\overline{\langle f_2 | f_1 \rangle}_{\mathrm{KG}} = \langle f_1 | f_2 \rangle_{\mathrm{KG}}$.
- (4) It is not positive definite.

- For example, in Minkowski space $\exp(\pm ik \cdot x)$ are two solutions on the mass hyperboloid $k^2 + m^2 = 0$, that is either on the positive shell $\mathbf{M}_+$ on which $k^0 = \omega = (\mathbf{k}^2 + m^2)^{1/2}$ or negative shell $\mathbf{M}_-$ on which $k^0 = -\omega$.

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- Their KG scalar products are

$$\langle e^{ik\cdot x}|e^{ik'\cdot x}\rangle_{\mathrm{KG}} = (2\pi)^3 2\omega\delta^3(\mathbf{k} - \mathbf{k}'). \tag{9}$$

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- Although plane wave solutions do not have finite norm, we can construct solutions of finite KG-norm from them: For each element $\psi(\mathbf{k}) \in \mathbf{L}_2(\mathbf{M}_+)$

$$f_\pm(x) = \int_{M_+} e^{\pm ik\cdot x} \psi(\mathbf{k})\, d\mu(\mathbf{k}),\ d\mu(\mathbf{k}) = \frac{d^3k}{(2\pi)^3 2\omega}. \tag{12}$$

- $f_{\pm}(x)$ are solutions of KG equation having finite KG-norm since $\langle f_{\pm}|g_{\pm}\rangle_{\mathrm{KG}} = \pm\langle\psi|\phi\rangle$ where $\langle\psi|\phi\rangle$ is the standard $\mathbf{L}_2(\mathbf{M}_+)$ scalar product and $\langle f_+|g_-\rangle_{\mathrm{KG}} = 0$.

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- $f_\pm(x)$ are called the positive and negative frequency solutions of the KG equation.
- The same calculations show that if $f(x)$ is a positive frequency solution then its complex conjugate $\overline{f(x)}$ is a negative frequency solution.
- So a general real solution of KG-equation is

$$\phi(x) = \sum \alpha_i f_i(x) + \overline{\alpha_i f_i(x)} \tag{13}$$

where f_i is the positive frequency solution associated with a basis $e_i(\mathbf{k})$ of the 1-particle Hilbert space $\mathbf{H}$ and α_i are some complex numbers. Since each solution is distribution valued, so is ϕ.

- The signs of $\langle f_{\pm}|g_{\pm}\rangle$ depend on our choice of $\epsilon_{0123} = -1$ and Hodge-star operation but the relative sign do not. A different choice will exchange the positive and negative frequency solutions.

- The signs of $\langle f_\pm | g_\pm \rangle$ depend on our choice of $\epsilon_{0123} = -1$ and Hodge-star operation but the relative sign do not. A different choice will exchange the positive and negative frequency solutions.
- The real scalar field operator is defined as follows: A real classical field is $\phi = \sum \alpha_i f_i(x) + \overline{\alpha_i f_i(x)}$. The complex number α_i carrying the label of the state $e_i(\mathbf{k})$ is elevated to the operator a_i. Similarly $\overline{\alpha_i}$ is elevated to the operator $a_i^\dagger$. So the hermitian scalar field operator is the sum $\phi(x) = \sum_i f_i(x) a_i + \overline{f_i(x)} a_i^\dagger$. Expanding the solutions,

$$\phi(x) = \sum_i \int_{M_+} \left(e^{ik\cdot x} e_i(\mathbf{k}) a_i + e^{-ik\cdot x} \overline{e_i(\mathbf{k})} a_i^\dagger \right) d\mu(\mathbf{k}) \tag{14}$$

In text books, $a(\mathbf{k}) = \sum_i e_i(\mathbf{k}) a_i$. However, $[a(\mathbf{k}), a^\dagger(\mathbf{k}')] = \sum_i e_i(\mathbf{k}) \overline{e_i(\mathbf{k}')} = (2\pi)^3 2\omega(\mathbf{k}) \delta^3(\mathbf{k} - \mathbf{k}')$.

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- They considered a scalar field ϕ_{in} in the far past and a field ϕ_{out} in the far future when all interactions are turned-off and solutions that interpolates between these fields.

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- Suppose the two fields are

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\begin{aligned}
\phi_{\text{in}}(x) &= \sum G_i(x) a_i + \overline{G_i(x)} a_i^\dagger, \\
\phi_{\text{out}}(x) &= \sum H_i(x) b_i + \overline{H_i(x)} b_i^\dagger
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- Some scattering operator S relates the two fields $S\phi_{\rm in}S^{-1} = \phi_{\rm out}$. This implies

$$Sa_iS^{-1} = \sum_j \langle G_i|H_j\rangle_{\rm KG} b_j + \langle G_i|\overline{H_j}\rangle_{\rm KG} b_j^\dagger. \tag{16}$$

- Now suppose in the far past H_i decomposes into a positive and a negative frequency parts as follows: $H_i = G_i' + \overline{G_i''}$. So while G_i is uniquely associated with the state $e_i \in \mathbf{H}_{\rm in}$, we suppose G_i' is associated with the state $A_{ij}e_j$ and G_i'' is associated with the state $\overline{B_{ij}}e_j$, where A_{ij}, B_{ij} are the Bogoliubov coefficients.

- Now suppose in the far past H_i decomposes into a positive and a negative frequency parts as follows: $H_i = G'_i + \overline{G''_i}$. So while G_i is uniquely associated with the state $e_i \in \mathbf{H}_{\rm in}$, we suppose G'_i is associated with the state $A_{ij}e_j$ and G''_i is associated with the state $\overline{B_{ij}}e_j$, where A_{ij}, B_{ij} are the Bogoliubov coefficients.
- Since H_i is uniquely associated with a state $\tilde{e}_i \in \mathbf{H}_{\rm out}$ in the out orthonormal basis, $\langle H_i|H_j\rangle_{\rm KG} = \langle \tilde{e}_i|\tilde{e}_j\rangle = \delta_{ij}$. So using $\langle \overline{G_i}|\overline{G_j}\rangle_{\rm KG} = -\langle e_j|e_i\rangle$ we get,

$$\begin{aligned} \delta_{ij} = \langle H_i|H_j\rangle_{\rm KG} &= \langle G'_i|G'_j\rangle_{\rm KG} + \langle \overline{G''_i}|\overline{G''_j}\rangle_{\rm KG} \\ &= \langle A_{ir}e_r|A_{js}e_s\rangle - \langle \overline{B_{js}}e_s|\overline{B_{ir}}e_r\rangle = (\overline{A}A^T - \overline{B}B^T)_{ij}, \end{aligned} \tag{17}$$

that is, $\overline{A}A^T - \overline{B}B^T = I$.

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that is, $\overline{A}A^T - \overline{B}B^T = I$.
- Similarly, we get other relations.

- Similarly, supposing that in the far future G_i decomposes into a positive and a negative frequency parts $G_i = H_i' + \overline{H_i''}$ and while H_i is uniquely associated with the state $\tilde{e}_i \in \mathbf{H}_{\text{out}}$, H_i' is associated with the state $C_{ij}\tilde{e}_j$ and H_i'' is associated with the state $\overline{D_{ij}}\tilde{e}_j$ where C_{ij}, D_{ij} are the Bogoliubov coefficients.

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- The independent relations among all the Bogoliubov coefficients can be re-written as

$$AA^\dagger - BB^\dagger = I, \quad AB^T = BA^T, \quad A^\dagger = C, \tag{18}$$

$$CC^\dagger - DD^\dagger = I, \quad CD^T = DC^T, \quad B^\dagger = -\overline{D}. \tag{19}$$

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- Using these relations we get $SaS^{-1} = A^T b + B^\dagger b^\dagger$. So if we consider a vacuum state $\Psi_0 = (\psi_0, 0, 0, ...) \in \mathbf{H}_{\rm in}$ then its image state $S\Psi_0 \in \mathbf{H}_{\rm out}$ must satisfy the constraint

$$SaS^{-1}S\Psi_0 = Sa\Psi_0 = 0 = (A^T b + B^\dagger b^\dagger)\Psi, \tag{20}$$

which in terms of C, D takes the form $\overline{C}b\Psi = \overline{D}b^\dagger\Psi$.

- On an arbitrary Fock state, it gives

$$\overline{C}_{ij}\Big(\tilde{\psi}_j,\ \tilde{\psi}_{jk}\tilde{e}_k,\ \frac{1}{2!}\tilde{\psi}_{jkl}\tilde{e}_k \otimes \tilde{e}_l, \ldots\Big)$$
$$= \overline{D}_{ij}\Big(0,\ \tilde{\psi}_0\tilde{e}_j,\ \tilde{\psi}_k\tilde{e}_{(j} \otimes \tilde{e}_{k)},\ \frac{1}{2!}\tilde{\psi}_{kl}\tilde{e}_{(j} \otimes \tilde{e}_k \otimes \tilde{e}_{l)}, \ldots\Big). \quad (21)$$

Since C is one-to-one, its inverse exists. Hence this constraint implies $\tilde{\psi}_i = \tilde{\psi}_{ijk} = \cdots = 0$, that is Ψ may contain only even particle states. This means Ψ is populated with particles created in pairs.

- On an arbitrary Fock state, it gives

$$\overline{C}_{ij}\Big(\tilde\psi_j,\ \tilde\psi_{jk}\tilde e_k,\ \frac{1}{2!}\tilde\psi_{jkl}\tilde e_k\otimes\tilde e_l,\ldots\Big) \\ =\overline{D}_{ij}\Big(0,\ \tilde\psi_0\tilde e_j,\ \tilde\psi_k\tilde e_{(j}\otimes\tilde e_{k)},\ \frac{1}{2!}\tilde\psi_{kl}\tilde e_{(j}\otimes\tilde e_k\otimes\tilde e_{l)},\ldots\Big). \qquad (21)$$

Since C is one-to-one, its inverse exists. Hence this constraint implies $\tilde\psi_i=\tilde\psi_{ijk}=\cdots=0$, that is Ψ may contain only even particle states. This means Ψ is populated with particles created in pairs.

- The image state $S\Psi_0$ measures a total number of particles

$$\langle S\Psi_0|b_i^\dagger b_i S\Psi_0\rangle=\mathrm{Tr}(BB^\dagger) \qquad (22)$$

where in the second step we have used $S^\dagger=S^{-1}$, that is S-matrix is unitary. The total number of particles is finite iff B is a trace-class operator.

- Suppose a massless scalar test field ϕ interacts with gravity when some matter collapses spherically to form an event horizon such that in the far past and future the spacetime is flat. At future null infinity a positive frequency solution is $H_\omega \sim \exp(-i\omega u)/r$. We extrapolate this solution to past null infinity to see whether we get a $\overline{G''}$.

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- The null ray does not hit the collapsing matter and gets reflected or absorbed by it. It can take the proposed path only at late times when most of the matter had already crossed the horizon and the ray is not affected by the collapsing matter.
- The null ray stays outside the event horizon. Since the Kruskal null coordinates are finite close to the event horizon, we should re-express the solution in Kruskal null coordinate $U = -\exp(-\kappa u)$ where κ is the surface gravity of the horizon.

- The KG norm of the positive/negative frequency solutions are

$$\begin{aligned}
&\langle \frac{1}{r} e^{-i\omega u} | \frac{1}{r} e^{-i\omega' u} \rangle_{\mathrm{KG}} = -\langle \frac{1}{r} e^{i\omega u} | \frac{1}{r} e^{i\omega' u} \rangle_{\mathrm{KG}} = (4\pi)^2 \omega \delta(\omega - \omega'), \\
&\langle \frac{1}{r} e^{i\omega v} | \frac{1}{r} e^{i\omega' v} \rangle_{\mathrm{KG}} = -\langle \frac{1}{r} e^{-i\omega v} | \frac{1}{r} e^{-i\omega' v} \rangle_{\mathrm{KG}} = (4\pi)^2 \omega \delta(\omega - \omega'), \\
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- This gives the map between the Hilbert space and positive frequency solutions

$$H_k(x) = \int_0^\infty \frac{\exp(-i\omega u)}{r} \frac{L_k(\omega l)}{k!} \sqrt{\omega l} e^{-\omega l/2} \frac{d\omega}{4\pi\omega} \quad (24)$$

where $\exp(-x/2)L_k(x)/k!$, $k = 0, 1, 2, ...$, are the orthnormalized Laguerre polynomials in $\mathbf{L}_2(0, \infty)$ and l is some arbitrary length scale. By construction, H_k are orthonormal in the KG-norm.

- So $H_\omega \sim \frac{1}{r}(-U)^{i\omega/\kappa}$. In the past null infinity $|U|$ becomes equal to $|v|$. Assuming the last ray from future reaching past along the event horizon is emitted at $v = 0$, the positive frequency solution of future extrapolated to past is $\frac{1}{r}(-v)^{i\omega/\kappa}$. On past the positive/negative frequency solutions are $\exp(\pm i\omega v)$ respectively.

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- The positive/negative frequency parts of $\frac{1}{r}(-v)^{i\omega/\kappa}$ give $A_{\omega\omega'}, B_{\omega\omega'}$:

$$A_{ks} = \langle G_s | H_k \rangle_{\mathrm{KG}} = \langle e_s | A e_k \rangle = \int_0^\infty d\omega d\omega' \langle e_s | \omega' \rangle A_{\omega\omega'} \langle \omega | e_k \rangle$$

$$B_{ks} = -\langle \overline{G_s} | H_k \rangle_{\mathrm{KG}} = \langle e_s | B e_k \rangle = \int_0^\infty d\omega d\omega' \langle e_s | \omega' \rangle B_{\omega\omega'} \langle \omega | e_k \rangle$$

where $\langle e_s | \omega \rangle = (L_s(\omega l)/s!)\sqrt{l}\exp(-\omega l/2)$. Calculating the KG-norms, we get

$$A_{\omega\omega'} = -\frac{1}{2\pi}\sqrt{\frac{\omega'}{\omega}}\frac{\Gamma(1+i\omega/\kappa)}{(-i\omega')^{1+i\omega/\kappa}}, \; B_{\omega\omega'} = \frac{1}{2\pi}\sqrt{\frac{\omega'}{\omega}}\frac{\Gamma(1+i\omega/\kappa)}{(i\omega')^{1+i\omega/\kappa}}. \quad (25)$$

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- Finally, the number of particles with frequency ω is obtained from the relation $(AA^\dagger - BB^\dagger)_{\omega\omega} = 1$

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- Do local calculations exist that do not involve mapping U, V coordinates to u, v?
- In a spherically symmetric collapse the metric is regular at the horizon in appropriate coordinates,

$$ds^2 = -\alpha^2 dUdV + r_s^2 d\Omega \tag{27}$$

  where α is a constant.

- The plane S-wave solutions are $\exp(-i\omega U/\kappa)$ or $\exp(i\omega V/\kappa)$, which are positive frequency on constant $T = \alpha(U+V)/2$ slices. However, the positive frequency eigenmodes wrt the timelike Killing vector field $i\kappa(-U\partial_U + V\partial_V)$ are $U^{i\omega/\kappa}$ or $V^{-i\omega/\kappa}$.

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- The KG-norms on constant T slices are

$$\begin{aligned}
\langle \frac{1}{r_s} e^{-i\omega U/\kappa} | \frac{1}{r_s} e^{-i\omega' U/\kappa} \rangle_{\rm KG} &= -\langle \frac{1}{r_s} e^{i\omega U/\kappa} | \frac{1}{r_s} e^{i\omega' U/\kappa} \rangle_{\rm KG} \\
&= (4\pi)^2 \omega \delta(\omega - \omega') \qquad (28) \\
\langle \frac{1}{r_s} e^{i\omega V/\kappa} | \frac{1}{r_s} e^{i\omega' V/\kappa} \rangle_{\rm KG} &= -\langle \frac{1}{r_s} e^{-i\omega V/\kappa} | \frac{1}{r_s} e^{-i\omega' V/\kappa} \rangle_{\rm KG} \\
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&= 0. \qquad (30)
\end{aligned}$$

- Similarly, the KG-norms of $(-U)^{i\omega/\kappa}$ and $V^{-i\omega/\kappa}$ on constant T slices

$$\begin{aligned}
\langle \frac{1}{r_s}(-U)^{i\omega/\kappa} | \frac{1}{r_s}(-U)^{i\omega'/\kappa} \rangle_{\mathrm{KG}} &= -\langle \frac{1}{r_s}(-U)^{-i\omega/\kappa} | \frac{1}{r_s}(-U)^{-i\omega'/\kappa} \rangle_{\mathrm{KG}} \\
&= (4\pi)^2 \omega \delta(\omega - \omega') \qquad (31) \\
\langle \frac{1}{r_s} V^{-i\omega/\kappa} | \frac{1}{r_s} V^{-i\omega'/\kappa} \rangle_{\mathrm{KG}} &= -\langle \frac{1}{r_s} V^{i\omega/\kappa} | \frac{1}{r_s} V^{i\omega'/\kappa} \rangle_{\mathrm{KG}} \\
&= (4\pi)^2 \omega \delta(\omega - \omega') \qquad (32) \\
\langle \frac{1}{r_s}(-U)^{i\omega/\kappa} | \frac{1}{r_s}(-U)^{-i\omega'/\kappa} \rangle_{\mathrm{KG}} &= \langle \frac{1}{r_s} V^{-i\omega/\kappa} | \frac{1}{r_s} V^{i\omega'/\kappa} \rangle_{\mathrm{KG}} \\
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&= (4\pi)^2\omega\delta(\omega-\omega') \qquad (31) \\
\langle\frac{1}{r_s}V^{-i\omega/\kappa}|\frac{1}{r_s}V^{-i\omega'/\kappa}\rangle_{\mathrm{KG}} &= -\langle\frac{1}{r_s}V^{i\omega/\kappa}|\frac{1}{r_s}V^{i\omega'/\kappa}\rangle_{\mathrm{KG}} \\
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&= 0. \qquad (33)
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- Because of these norms, the mapping of the positive frequency solutions to the Hilbert space remain the same as before. So we can construct both solutions H_k and G_k, orthonormal in KG-norm.

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- It is purely a local calculation except that one has to consider the Killing vector.
- It is not completely free of ambiguities because one can introduce more than one regular coordinates close to the horizon. However, I believe that the result won't change in other regular coordinates.